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AUTHOR MCALOON, MARY DE LOURDES
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INSTITUTION MICHIGAN UNIV., ANN ARBOR.
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ABSTRACT

THIS IS ONE OF A SERIES OF UNITS INTENDED FOR BOTH PRESERVICE AND INSERVICE ELEMENTARY SCHOOL TEACHERS TO SATISFY A NEED FOR MATERIALS ON "NEW MATHEMATICS" PROGRAMS WHICH (1) ARE READABLE ON A SELF BASIS OR WITH MINIMAL INSTRUCTION, (2) SHOW THE PEDAGOGICAL OBJECTIVES AND USES OF SUCH MATHEMATICAL STRUCTURAL IDEAS AS THE FIELD AXIOMS, SETS, AND LOGIC, AND (3) RELATE MATHEMATICS TO THE "REAL WORLD," ITS APPLICATIONS, AND OTHER AREAS OF THE CURRICULUM. THIS UNIT EXPLORES WITH TEACHERS IDEAS ABOUT INFERENCE AS THE OPERATION BY WHICH NEW KNOWLEDGE IS OBTAINED BY LOGICAL IMPLICATIONS FROM THE KNOWN. INCLUDED AMONG THESE IDEAS ARE CONCEPTS ON SENTENCE SENSE, VARIABLES, PLACEHOLDERS, CONNECTIVES, INFERENCES FROM NEGATIONS, CONTRADICTIONS, IMPLICATIONS, AND TRUTH VALUES OF IMPLICATIONS. (RP)

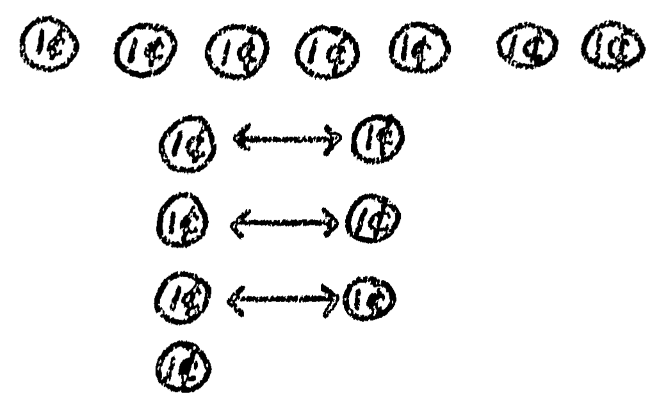
The University of Michigan
U.S.O.E. Elementary School Materials Project
Sr. Mary de Lourdes McAloon

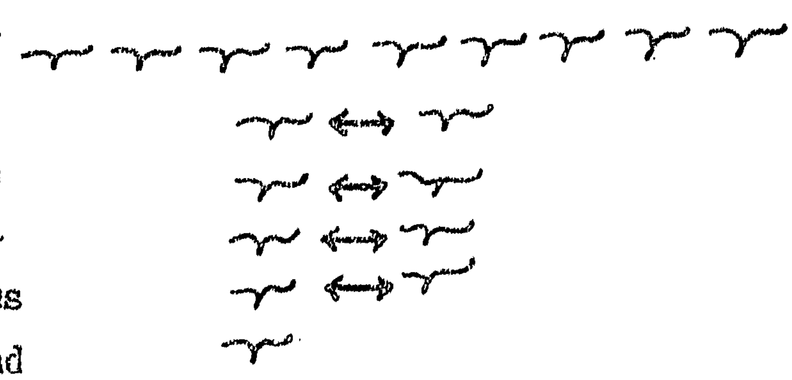
LOGICAL THINKING

1.1 Introductory Ideas

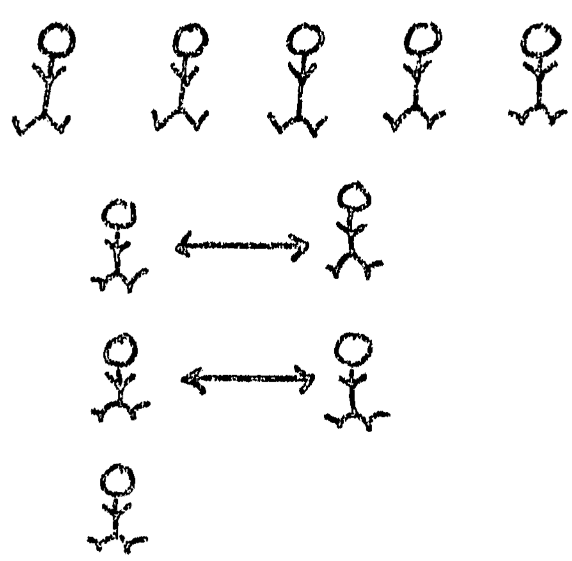
When the phrase logical thinking is used, a certain ambiguity is attached to it. To dispel this ambiguity and to escape being convicted of "loose language," we will consider inference as the operation by which the mind obtains new knowledge by drawing out the implications of what it already knows. This broader view is in contrast to its precise logical meaning; a process of reasoning by which the mind proceeds from one or more propositions to other propositions seen to be implied by the former.

A small child is not able to "glean" all the information from a statement which an adult might infer. For example, a child may be able to infer that 7, 9, or 5 are odd numbers because he has paired the elements in sets of 7, 9, or 5 and noted that an extra or odd one was left over. (See diagrams to the right.) Then, as he learns division, he can infer that when an odd number is divided by 2, there is a remainder of one. The idea of "left over," while not easy, is fairly intuitive in the early stages. As he learns the new operation of division, the relation between "left over" and "remainder" becomes more forceful to him. As the child's background increases and he matures, he will be able to make other inferences.





However, inference can be a precarious process. For example, if two or more premises are given, a learner could quite possibly have more than one choice of a conclusion. This conclusion would



depend not only on logical but psychological processes as well. A child's experience, as well as that of the teacher, is often a determining factor in the kinds of inferences that are made and the types of conclusions drawn from certain premises. As an example of this, note the different reactions made by adults, teen-agers, or children to identical statements. Therefore, it is imperative throughout this booklet that teachers be aware of the factors not strictly logical or deductive in nature which influence inferences.

In the case of everyday reasoning which is, in a real sense, a form of inference, there has been considerable difficulty attributed to the correct use of quantifiers such as some, all, or none. To young children, all does admit exceptions. "Everyone is out--except my daddy." "All at home think I am cute--except my mother." Careful observation will disclose that this type of "all with exceptions" is quite in vogue in colloquial usage. A teacher cannot forget this situation--an apparent paradox that seems, at times, insolvable.

However, rather than ignore the problem, it is wise to face it from the beginning somewhat as one might face a translating problem. For example, a diplomat who speaks in the UN expects the interpreters, in translating French into English, to repeat or indicate in the new language exactly or as precisely as possible what was intended in the original. There is to be no ad-libbing here. Contrasted with this, a person might want someone else to translate ideas into another language. This translator has a more difficult job; he has the added obligation of being logical, consistent, and "not loose" in his language. The interpreter for the diplomat, on the contrary, need only repeat, even the illogical, the inconsistent--he was actually nothing more than a "linguistic parrot," as it were.

Therefore, we must approach the matter of thinking from a dual role, namely, while trying to improve the logical, we must be mindful of the psychological. We try, in our own formulation of thoughts and the teaching of mathematical ideas to reflect consistency, but do admit in general transactions that not everyone engages precisely in this kind of thinking. As in the example above, all does not mean without exception because we are using it in a colloquial sense. There is no dichotomy; it is a matter of viewpoint and usage.

In the exercises as well as in the explanations there will be an awareness of both situations. Problems and sentences used to clarify ideas, although

primarily directed at the teacher, could be used in a classroom situation with some slight adaptation of vocabulary. Special sections on the development of inferential questioning techniques are included throughout the booklet.

2.1 Stages of Cognitive Development in Children

Teachers must be cognizant of the stages of cognitive development in the child while attempting to structure mathematical ideas in a logical way and to promote the development of understanding and use of more precise language and reasoning processes. Piaget, in his works, listed these generalizations concerning cognitive development:

1. There is absolute continuity in all developmental processes.
2. Development proceeds through a continuous process of generalizations and differentiation.
3. This continuity is achieved by a continuous unfolding. Each level of development finds its roots in a previous phase and continues into the following one.
4. Each phase entails a repetition of processes of the previous level in a different form of organization (schema). Previous behavior patterns are sensed as inferior and become part of the new superior level.
5. Differences in organizational pattern create a hierarchy of experience and action.
6. Individuals achieve different levels within the hierarchy, although there is in the brain of each individual the possibility for all these developments, but they are not all realized.

(Maier, Three Theories of Child Development, 92)

Piaget claims that cognitive development is dependent upon the construction of mental operations. An operation is considered to be an internalized action which becomes reversible, that is, it can be carried out in both directions and linked up with others. There are four main stages in the construction of operations which extend over the period from birth to maturity. Briefly, these stages can be described as:

1) SENSORI-MOTOR period: (0 to 2 years): Initial phase during which the child can only perform motor actions, although these actions display some features of intelligence. For example, the child will draw a blanket toward itself in order to obtain an object placed on it. These elementary operations correspond to the problem-solving abilities of sub-human animals. An interesting aspect of this stage is the one involving the construction of the permanent object, an object which continues to exist beyond the limits of the perceptual field.

That is, the child realizes that the object still exists although he cannot see it. At first, the infant gives up any attempt to find objects as soon as they are concealed. Gradually, he learns to attempt the discovery of these.

2) PRE-OPERATIONAL THOUGHT: (2 to 7 years): The child begins to communicate by language. Activity is dominated by "symbolic play" which imitates and represents what he has seen others do. In the realm of make-believe or role-playing, the child differentiates between the symbol and that which is symbolized; i.e. he knows that the doll is not really a baby or that he is not really an astronaut.

3) INTUITIVE THOUGHT AND CONCRETE OPERATIONS: (7 to 11 years): Concepts which are internalized actions are called "operations." As the child interacts repeatedly with things and people, his central processes become more and more autonomous. Gradually he learns to group action-images into systems which permit clarifying, ordering in series, remembering. Considered here are concepts of

a) classes--grouping objects together which are recognized as similar, and

b) relations--ordering activities such as placing objects in a row in an order of increasing size.

4) FORMAL OPERATIONS: (11 or 12 to 14 or 15 years): The child begins to group or systematize his concrete operations (classifications, relations and serial orderings) and thereby to consider all possible combinations in each case. The child becomes essentially adult, can now operate with the form of an argument while ignoring its empirical content. With new-found thought structures, he need no longer confine his attention to existing reality. He becomes concerned socially with seeing the world as it might better be and enters the roles of critic and social reformer.

This brief look at the stages which Piaget considers crucial in the development of a child's thinking is meant to alert the elementary teacher to the need for a careful examination of the types of presentations appropriate for children and of expectations which are within reason. A child of six is not going to derive the same benefit from a lesson in geometry as a child of ten or twelve.

An individual's cognitive development tends to coincide with Piaget's ranges which in their preliminary presentation were age-bound. The order of succession is the important thing, and Piaget's stages are age-free in terms of their order of sequence.

2.2 Experience, An Important Factor

It is crucial to remember that one cannot expect a child of lower elementary level to have the subtle reasoning powers that one would expect from a college graduate. Nevertheless it is conceivable that young children can make simple inferences with the help of questioning techniques on the part of the teacher from a statement such as:

The door is locked.

i) Does this mean the door is closed?
ii) Is the door open?
iii) Does it have a key in it?
iv) How can you tell the door is locked?
v) Can you tell the door is locked by just looking at it?
vi) Do you have to try it?
vii) Could the door be locked but be opened?

Obviously, the sequence of questioning will get its order from the responses of the children. Then, too, children without any experience of locks and keys would not be able to understand the point of a discussion of this kind. The topic, the would-be educational experience, must be adapted not only to the age level of the child but to his experience and background also. What is often misinterpreted as failure in logical reasoning is merely a deficiency or lack of experience.

In English classes and even, in fact, since you have been three years old, you have been studying sentences. As you have grown older, you get more from a sentence. In an exchange, for example, between two small boys, suppose John says:

"Today is Monday."

Jimmy, his friend, will take the sentence for what it is worth, probably no more or no less. An adult might draw additional information from the simple statement, thinking that yesterday was Sunday, tomorrow is Tuesday, and so on. Jimmy, if something special happens to him on Monday, might think, "Swell, today my grandmother brings cookies to us."

Quite possibly the ability to make more inferences can be based on the previous experience. At any rate, it is an important concept which should pervade the consciousness of the teacher when analyzing any deficiencies of reasoning on the part of the child. If there is no experience, a statement could be just another collection of words put together for a conversational piece.

In order to infer, to gather additional information correctly from statements that people make, we must be concerned with truth and falsity as well as validity. It would be utter folly to try to elicit any implications from a statement such as:

"Abraham Lincoln is the first American astronaut."

To attempt to say that the United States has a space program solely on the basis of this sentence would not contribute a great deal and, indeed, would be building on a rather shakey foundation.

On the other hand, to say that:

"5 is greater than 2"

can indicate, because it is true, that:

a) there is some whole number needed to add to 2 to make it equal to 5,
b) 5 - 2 is greater than zero,
c) if you go to the left on an ordinary number line, the numbers are less than those on the right. 2 is to the left of 5, and
d) if you go to the right on an ordinary number line, the numbers are greater than those to the left. 5 is to the right of 2.

Reactions to this statement can be varied, depending not only upon the maturity of the student but also the "degree of exposure" to this type of number comparison.

3.1 Sentence Sense

Elementary mathematics is concerned with a special kind of mental activity, namely, getting information and ideas from sentences. However, first we must look at the structure and kinds of sentences which are appropriate for the drawing of inferences.

Consider the following examples:

1. Elizabeth is queen of England.
2. In beautiful Ohio.
3. Do you like candy?
4. He is taller than his father.

Which tell us something? Clearly only (1) and (4) give us a complete picture. Example (2) is not even a complete sentence although it might evoke some pleasant memories of one of the states.

Do you notice any difference between (1) and (4)?

Is (1) true or false?

Is (4) true or false?

How about (3)? Would it even make sense to ask whether (3) is true or false?

Example (1) is true. Why can't you tell whether (4) is true or false? If you replaced "he" with someone's name, could you tell whether the sentence is true or false? Obviously since (2) is not even a sentence, it cannot tell us much.

Therefore, it would seem that we are interested only in declarative sentences, sentences that tell us something. So we define: <u>a sentence that can be judged true or false is called a statement</u>.

In elementary school mathematics the child learns at an early age about true and false statements. For example, in the primary grades when a child fills a placeholder correctly, that is, when there is a number sentence properly completed, he is participating in the construction of true-false statements.

Put the correct numeral in the placeholder in order to make a true statement in each of the following:

1. $6 + \square = 8$
2. $4 - 1 = \square$
3. $\square + 0 = \square$

Place the correct symbol, >, <, or = on the _____ to make a true number sentence:

4. 6 + 2 _____ 7
5. 0 _____ 6 - 6
6. 5 _____ 8 - 1

None of the sentences 1 - 3 can be judged true or false until the child supplies some numbers. These are called OPEN SENTENCES, that is, a definite replacement is needed before you can indicate truth or falsity.

Particularly in 4 - 6, the symbols and relations cannot even be considered sentences because the "mathematical verbs" are not there. There is enough evidence to suspect, however, that with some additional mental activity, the statements will be forthcoming. Implied, therefore, is a sentence. At any rate, both kinds of activities are an integral part of elementary mathematics.

EXERCISE A

Indicate whether the following are sentences. If they are, tell whether they are true or false or are open sentences.

1. For every three minutes of the day.
2. Every time a man does something, he uses up energy.
3. $3 + 4 = 6$.
4. He was an American writer.
5. Figure A shows a picture of a square.
6. $N + 5 = 8$.
7. Cars that are red and shiny with drivers who are eager and expert.
8. $8 \div 2$.
9. Figure B shows a picture of a square region.
10. $5 - 3$, $3 + 0$, $6 \div 2$, $1 + 2$.
11. $x + x = 2x$.
12. Fish are lighter than.
13. This number is greater than two.
14. $\square - 1 = 9$.
15. All boys are handsome.
16. $3 + 5 > 2 + 4$.

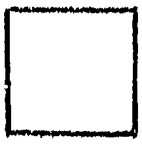


Figure A.

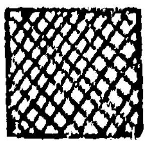


Figure B.

Inferential Questioning Techniques

Example 16 is, of course, true. Some inferences could be made from this or rather we could build up some techniques useful in having children make inferences. In the beginning, they will need more help but gradually should be able to ask themselves questions, look for different aspects.

A sequence such as the following could be used:

a) If you add two whole numbers, is the sum always greater than either of the numbers?

b) Depending, of course, upon the responses to the initial query, but supposing that the child has answered correctly, you continue the questions: If you say "No," to (a), give examples.

c) Suppose neither number is a zero, if you add two whole numbers, do you always get a sum greater than either of the numbers?

d) Is $3 > 2$?

e) Is $5 > 4$?

f) In #16, is it necessary to find the sums on each side of the inequality symbol before you make a decision? (Some might not agree, might think performing the operation is absolutely essential but a close scrutiny will indicate that while this may be done, it does not have to be. Necessary means that a person cannot find the answer without this technique.)

g) If $3 > 2$ and $5 > 4$ and you add $3 + 5$, do you suppose that it will be greater than $2 + 4$?

3.2 Variables, Placeholders

In sentences such as 4 and 14 of the Exercise A, the pronoun "he" and the placeholder hinder us from determining whether the sentence is true or false. The "he" in sentence 4 could be replaced by a person's name and the sentence would become a statement, capable of being judged true or false.

"He" in sentence 4 and $\square$ in sentence 14 are called variables or place-holders because there are several names or numerals which we could put in the place of "he" or $\square$.

Obviously the use of the pronoun "he" limits the replacement set to all males. Sometimes we can tell the set from which to choose and sometimes we cannot. For example, in the following:

1. $2 \times 6 = \square$
2. $8 \times 2 = \square$
3. $9 \times 2 = \square$
4. $2 \times 3 = \square$
5. $2 \times 5 = \square$

the replacements must be even numbers because when we multiply any whole number by 2, the product is even.

The situation is different in the following:

6. $\square + 2 = 7$
7. $3 \times \square = 21$
8. $\square \div 1 = 4$
9. $\square - 2 = 6.$

Both odd and even numbers can be put in the $\square$'s. On the other hand, it is possible to construct exercises such that after a child replaces the variables to make true statements he can go back and infer from his replacements a general class of numbers.

10. $\square + 4 = 9$

11. $7 \times 3 = \square$

12. $\square + \square = 10$. (Since we used the same type of placeholder for each, they must be the same. Otherwise it would be indicated as $\square + \triangle = 10$.)

13. $15 \div 3 = \square$

14. $\square \times 2 = 14$

15. $\square \times \square = 25$

In each of the sentences above, the replacement is an odd number. This cannot be discerned until the children have finished all of the problems and been urged, either by questioning at strategic intervals or by habitual reexamination of what they have done, to look carefully at what is in front of them. Primary youngsters might not be able to make any inferences at this stage regarding the replacement set unless a teacher asks some leading questions. However, middle and upper elementary students might not only recognize the pattern for odd replacements but should be able to make some additional generalizations such as:

Odd + odd = even.

Even + odd = odd.

The product of two odd numbers is odd but the sum of two odd numbers is even. Notice the distinction in (11) and (12). In (11) the two odd numbers are there, but in (12) one has to look carefully at the one side and then the other to determine whether or not there are to be two even numbers or two odd numbers.

In (15) relatively the same idea occurs. One has to look carefully not only at the variables but also at the product because the product of two even numbers is even and an additional step or scrutiny is necessary.

<u>4.1 Connectives or (Junction Functions?)</u>

Children mature in the recognition of patterns of numbers such as in the series:

a) $1 + 3 = 4$

b) $1 + 3 + 5 = 9$

c) $1 + 3 + 5 + 7 = 16$

d) $1 + 3 + 5 + 7 + 9 = 25$

and so on.

Let us look at these sums carefully.

How many odd numbers are added in (a)?

How many odd numbers are added in (b)?

How many odd numbers are added in (c)?

How many odd numbers are added in (d)?

If you made a series (e), how many odd numbers would you have?

What would its sum be?

Write down the sums in a sequence as such:

4, 9, 16, 25, 36

Is there anything special about these numbers?

Now rewrite the entire series in the following way:

a) $1 + 3 = 4 = 2^2$

b) $1 + 3 + 5 = 9 = 3^2$

c) $1 + 3 + 5 + 7 = 16 = 4^2$

d) $1 + 3 + 5 + 7 + 9 = 25 = 5^2$

e) $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

What would be the sum of the first n odd numbers?

As children grow older, the problems and pattern recognition exercises can be made more difficult. Then, too, children begin to see the pattern or logical structure of statements; that is, the way in which ideas or relations are tied together by such connecting words as: and, or, not, if . . . then, if and only if. Perhaps in the beginning, they do not realize the full import of what they are saying, but even a pre-school child will say,

"If you do this, then I will give you a cookie."

While the child does not know yet what he is getting himself in for when he uses implications, nevertheless he has some intuitive idea that a condition of some sort or other is present in his statement. It sometimes amounts to, "Well, you didn't do it, so I am not giving you a cookie." Enough about this for the moment. Let us examine carefully some simple ideas of conjunction and disjunction.

4.2 The Conjunctive "AND"

When we combine two sentences with the word "and", we call the resulting sentence a CONJUNCTION.

"Smith is a tailor. Jones is a sailor."

"Smith is a tailor and Jones is a sailor" is a conjunction.

As you studied simple statements, you noted that they did not have any other statements as component parts, but compound statements do have another statement as a component part. Therefore, a CONJUNCTION is a compound proposition in which the two components are connected by the word "and" or some other word that conveys the same meaning. Each of the component parts may be called a conjunct, but the conjunctive proposition must be regarded as a single proposition.

Examples:

1) Five is greater than two and five is less than eight.
2) Three is an even number and two divides ten.
3) The two lines g and h are perpendicular and they are parallel to line p.

What does sentence (1) assert? It claims that not only is the statement "Five is greater than two" true but also that "Five is less than eight" is true. Even more crucially, it asserts the truth of the conjuncts taken together. In sentence (2), obviously, three is not an even number, so one of the conjuncts is false. This conjunctive proposition (2) itself must be false even though the other conjunct "Two divides ten" is true. We can generalize:

A conjunction is true if and only if both component statements are true.

i) What is the minimum number of propositions, which, if false, makes a conjunction false?

ii) Is a conjunction which has two false conjuncts true or false?

Consider the following propositions which are conjunctions:

4) Triangles and circles are simple closed curves but triangles are not circles.
5) Although all polygons are simple closed curves, not all simple closed curves are polygons.
6) Although every integer can be viewed as a rational number, not every rational number is an integer.

Using symbols to represent statements sometimes helps clarify the study of statements and logic. Thus in (1) we could let f represent the statement, "Five is greater than two" and e represent the statement, "Five is less than eight." Thus we can represent (1) symbolically by $f \wedge e$ where $\wedge$ symbolizes the conjunction "and."

In Example 4 we can symbolize the conjunction as follows:

c = Triangles and circles are simple closed curves.

t = Triangles are not circles.

$$c \wedge t$$

Note that "but" connects the two propositions since it stands between two portions of the sentence each of which could stand alone as a sentence. Logically, "but" means the same as "and," even though psychologically there might be a distinction. You might feel that "but" says more than "and." You know that when you hear, "Yes, but . . ." instead of "Yes, and . . ." you do have a different reaction. This is a psychological significance, not a logical one. By that, we mean that there is no case where "p but q" would differ in truth and falsity from "p and q." It is imperative that we realize this when looking at these basic structures. Often there is a shift from the logical to the psychological and back again in this discussion. Even though the emphasis shifts from one to the other at various times, neither can be neglected in thinking about and working with inferences.

In (5), "although" carries the same logical significance as "and" whatever its psychological character. Regarding immediate inference, not much can be said about conjunctions except that when we assert that a conjunction is true, we know that each of its components must be true. If we state that a conjunction is false, we note that at least one conjunct is false.

In a sentence such as "Two and three are divisors of six," the word <u>and</u> is used to connect words, not propositions. Therefore, this sentence is <u>not</u> considered a conjunction. However, it could be written as a conjunction: "Two is a divisor of six and three is a divisor of six." Notice that neither sentence is true unless <u>both</u> two and three divide six.

Just as simple inferences can be drawn in elementary mathematics from simple statements, so also the conjunction is often used, (expressed or implied), in the primary grades and inferences required. Even in the first grade a child is asked to consider the idea of betweenness, and exercises of the following kind are given to him to decide which number is between, for example, 5 and 7.

5, _____ , 7

Six is between 5 and 7 because it is greater than five <u>and</u> it is less than 7. Eight is not between 5 and 7. True, it is greater than 5, but it is not less

than 7, so we cannot say that it has betweenness relative to 5 and 7. Six is between 2 and 10 because 6 is greater than 2 and 6 is less than 10. Can you say that 5 is between 6 and 8? Why not? It has to be greater than one of the numbers and less than the other. This is not so in the case of 5 and its relation to 6 or 8.

Whether or not a child actually verbalizes a conjunction is not important. The crucial point is that he cannot use the concept of betweenness without considering both elements or components of a conjunction, either expressed or implied. In the primary and lower elementary grades, this is often implied. Moreover, since the idea of betweenness is important in the development of counting, it is conceivable to think that a child who does not understand the notion of one number being greater than another and less than a third one, may not understand what he means when he says "comes after" or "comes before."

4.3 The Disjunctive "Or"

When we combine two sentences with the word "or," we call the resulting sentence a DISJUNCTION. In the sentences "Smith is a box maker" and "Jones is a caretaker," the combination "Smith is a box maker 'or' Jones is a caretaker" is a disjunction. As in the case of the conjunction, the disjunction is a compound proposition, but it must be regarded as a single proposition which may be either true or false.

It is the determination of truth and falsity of combinations of sentences that causes difficulty at times. Whenever we put sentences together, we have to determine the truth and falsity of an additional statement. That is, if we have "P and Q," we not only judge the truth and falsity of P separately, then Q separately, but of the combination. This is true for the disjunction "P or Q" also. In regard to determining the truth or falsity of conjunctions, it can be done effectively by asking the children to decide about several sample conjunctions. They agree quite readily that both components have to be true. This is not true in the case of disjunctions.

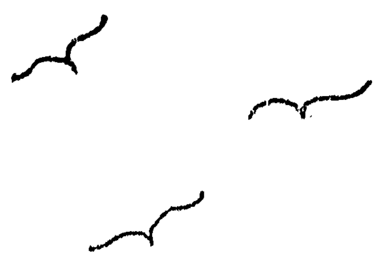
Children do not seem to be as readily able to decide about the truth or falsity of disjunctions because a little more is involved. There will be obvious disagreement and it is wise to foster it. Then, hopefully, the children will come to realize that since there is no intuitive agreement or that "everyone is

yelling at once," it is necessary to establish a rule or a definition. It amounts to distinguishing between the "exclusive" and "inclusive" use of <u>or</u>. For most present purposes, we are concerned with the inclusive use of <u>or</u>.

In the exclusive sense, a human being is either dead or alive. Not much reasoning is necessary to realize that a person cannot be both. Nor is much unrest present when children are deciding whether 7 is greater than 5 + 1 or whether 5 + 1 is greater than 7. Both cannot be true. They are exclusive; that is, if one is true, the other cannot be. For the exclusive use of <u>or</u>, then, it is fairly easy for children to determine the truth or falsity of a statement.

However, children use intuitively and "conveniently" both the exclusive and the inclusive <u>or</u> in everyday activities. Recall, "Either John or you has to do the dishes," or "Either John or you has to clean the room after school." This interpretation, unless the child is rare, will be exclusive and no doubt John will be the victim. On the other hand, "Mary may have a quarter or Susie may have a quarter," will be inclusive--that is, if Mary and Susie are good friends and there is enough available cash.

In the lower elementary grades and, in fact, even in kindergarten, children are asked to look at a picture such as the one at the right. Shown in the diagram is a picture of a set of birds. Mark the numeral which tells how many birds are in the set.



1 2 3

Obviously only one numeral can be marked.

While this is an implicit use of <u>or</u>, seldom does a child mark more than one. Granted, he might not mark the correct one, but he knows intuitively that not more than one number can express how many elements are in a set.

On the other hand, suppose that someone asks the question, "Can Johnny come to school by bike or by foot?" Quite conceivably, he can use both. He might come part way by bike and then walk the rest. While this is not a mathematical use, nevertheless it is a real part of the child's thinking.

Consider the following sequence of disjunctions. Can you tell which are true and which are false?

1. 4 is an even number or 6 is greater than 5.
2. May 30 is Memorial Day or Washington, D.C. is the sixth month of the year.
3. June is a day of the week or Tuesday is the sixth month of the year.
4. Lemons are sweet or Ohio is one of the 50 states.
5. John Smith, the man who used to live on the corner, is dead or he is alive.

Note that we can have several choices and if we use the reaction of the children as potently as possible, we can lead them to realize that some "consensus" is necessary. (1) is obviously true. (3) will be judged quite obviously false. Trouble and disagreement usually appear with such examples as (2), (4) and (5) where one of the components is not true. The first four use <u>or</u> in the inclusive sense while (5) uses it exclusively.

After some discussion, the children can decide, perhaps even vote (under the careful, "unbiased" steering of the teacher) that in a disjunction if at least one of the components is true, the disjunction is judged to be true. Therefore, we can state: <u>A disjunction is false if and only if both component statements are false.</u>

<u>4.4 Inferring Information from Disjunctive and Conjunctive Sentences</u>

Inferences can also be drawn from the use of the disjunction and the conjunction, and questioning children along these lines leads to much more insight into the use of conjunctive and disjunctive reasoning in mathematics. The following examples and questions which may be used with students will be instructive.

Consider the sentence, "Batman is on TV and he wears a mask." Obviously we have an example of a conjunction in which each of the component parts is true. Since we know definitely the replacement for <u>he</u>, we can determine the truth or falsity of the conjunction. In other words, what we actually mean, but are saying in accepted usage, is:

(1) <u>Batman is on TV and Batman wears a mask.</u>

i) Can both components be true?

ii) Is "Batman is on TV" true?

iii) Does Batman wear a mask?

iv) Is the conjunction true?

(2) Baseball is an American sport and golf is played with a baseball bat.

i) Is baseball an American sport?

ii) Is golf played (officially and legally) with a baseball bat?

iii) Would you say that one of the components of the conjunction is true?

iv) Is one of the components of the conjunction false?

v) Is the conjunction false?

(3) Boys like to play sports or children are always quiet.

i) Do boys like to play sports?

ii) Are children usually quiet?

iii) Are children always quiet?

(If there is any disagreement or discussion desired, do not hesitate to pause in the sequence of questioning. You can always pick it up again by repeating quickly the questions asked previously, so that the pattern of development is more easily discernible for the children.)

iv) Do you think that one of the components is true and one is false?

v) Do you think the disjunction is true or false?

vi) Would it be possible to say that one of these must be true and one must be false?

vii) How does this example differ from the one, "Seven is greater than 4 or 4 is greater than 7?"

(Here, quite possibly, a teacher might once again mention the inclusive and the exclusive use of or, using examples from the daily lives familiar to the children. Dead or alive are mutually exclusive properties despite the popularity of the colloquial expressions, "half-dead," or "half-alive." While admitting the existence and usefulness of colloquialisms, nevertheless, a teacher should help the children to realize the more precise meanings of mathematical or other technical terms. In order to get a broad interpretation of reasoning, a teacher must not forget that psychological aspects are important and even humor, at times, is an aid to making sequences appear more "sensible" and "consistent.")

(4) There are more letters in "OHIO" than there are in "MICHIGAN" and Texas is a northeastern state of the United States.

i) Name two components of this conjunction.

ii) Is "There are more letters in "OHIO" than there are in "MICHIGAN" " true or false?

iii) Is Texas a northeastern U.S. state?

iv) Would you say that both components are false?

v) Is the conjunction false?

vi) When would you consider a conjunction false?

(5) Seven is greater than 4 or 4 is greater than 7.

i) Can both of these be true?

ii) Does that mean that either one or the other has to be true?

iii) Is the disjunction true?

(6) Colorado is in the Rocky Mountains or California borders the Pacific Ocean.

i) Is "Colorado is in the Rocky Mountains" true or false?

ii) Does California border the Pacific Ocean?

iii) Then you consider both of these true? (This question will, quite obviously, depend upon the answers given to i) and ii). You will note that this is true for the other sequences also and if you vary your questions according to the answers you get on the "way along," good for you.)

iv) Would you consider then that the disjunction is true?

(7) 6 is less than 2 or 5 is greater than 8.

i) Is 6 less than 2?

ii) Is 5 greater than 8?

iii) Do you consider both components of this disjunction to be false?

iv) Then is the disjunction itself true or false?

In summary, a teacher could continue the sequence:

v) What is needed to make a conjunction true?

vi) What is needed to make a conjunction false?

vii) What is needed to make a disjunction true?

viii) What is needed to make a disjunction false?

ix) Do you think "more falsity" is needed to make a disjunction or a conjunction false?

x) Did you find it easier to discover when a conjunction is true or when a disjunction is true?

4.5 Looking Things Over

This might be a good opportunity for a teacher to retrace her steps and have the students go back to the original examples for the conjunction and change every and to or and then determine the truth and falsity of the new sentences. This is merely a suggestion so that the structuring of responses can be put into diagrammatic form or some kind of pattern to facilitate a child's seeing distinctions and similarities between the two ideas. Then, too, this involvement in being a partner in "inferential questioning" such as shown previously helps the student look for hidden observations. Hopefully he will learn to do this on his own; to learn how to learn.

The review could be organized as is shown below. The symbol "$\wedge$" is read "and"; the symbol "$\vee$" is read "or." Of course, the truth or falsity of each disjunction and conjunction should be discussed with the children.

Conjunction	Disjunction
Batman is on TV and Batman wears a mask.	Batman is on TV or Batman wears a mask.
Let: B = Batman is on TV. W = Batman wears a mask.	Let: B = Batman is on TV. W = Batman wears a mask.
Symbolically: $B \wedge W$	Symbolically: $B \vee W$
Baseball is an American sport and golf is played with a baseball bat.	Baseball is an American sport or golf is played with a baseball bat.

Let: B = Baseball is an American sport.
G = Golf is played with a baseball bat.

Symbolically: $B \wedge G$	Symbolically: $B \vee G$
Little Red Riding Hood is a witch and George Washington was an American president.	Little Red Riding Hood is a witch or George Washington was an American president.

Let: L = Little Red Riding Hood is a witch.
G = George Washington was an American president.

Symbolically: $L \wedge G$	Symbolically: $L \vee G$

There are more letters in "OHIO" than there are in "MICHIGAN" and Texas is a northeastern state of the United States.

There are more letters in "OHIO than there are in "MICHIGAN" or Texas is a northeastern state of the United States.

Let: M = There are more letters in "OHIO" than in "MICHIGAN."

T = Texas is a northeastern state of the United States.

Symbolically: $M \wedge T$

Symbolically: $M \vee T$

4.6 Truth Tables

In the previous examples we used some symbols to represent the entire statement or component parts. This is done for the purpose of simplicity, not complexity. In other words, symbols are introduced to make a teacher's life more simple, not more complicated. A rather striking example follows:

There are more letters in "OHIO" than there are in "MICHIGAN." False.

Texas is a northeastern state of the United States. False.

"There are more letters in "OHIO" than there are in "MICHIGAN" and Texas is a northeastern state of the United States" is a false conjunction because both components are false.

It is rather awkward at times for children (and for teachers) to look at a group of sentences and realize all possibilities of truth and falsity. Each component of a conjunction can be judged either true or false. Since there are two components, that means that there are four possibilities for truth values. Let us look at these in a diagram.

Use the symbols as indicated for the four examples of conjunctions and disjunctions in Section 4.5.

Conjunctions

1)

B	W	$B \wedge W$
T	T	T

2)

B	G	$B \wedge G$
T	F	F

3)

L	G	$L \wedge G$
F	T	F

4)

M	T	$M \wedge T$
F	F	F

Disjunctions

1)

B	W	$B \vee W$
T	T	T

2)

B	G	$B \vee G$
T	F	T

3)

L	G	$L \vee G$
F	T	T

4)

M	T	$M \vee T$
F	F	F

Even symbols can be put into complicated patterns, structures, or tables. Despite the fact that the above diagram does simplify the writing, it is not quite rid of all possible complexities. We can consider a statement, P, any statement which can be judged either true or false and another statement, Q, which can also be judged true or false. What are the possible combinations of truth or falsity of these two statements?

In order to utlizie the ideas of a truth table and to use some of the basic notions of all possibilities of true and false values, we can make an analogy with the idea of taking two coins, a nickel and a dime, and flipping these. What are the possibilities? Obviously, there are only two possibilities if we are flipping one coin, namely heads or tails. What about two of the coins?

Let N=nickel D=dime H=heads T=tails

The possibilities can be shown as follows:

Nickel	Dime	
H	H	On the flip, both coins turn up heads.
H	T	On the flip, the nickel turns up heads, while the dime turns up tails.
T	H	On the flip, the nickel turns up tails and the dime turns up heads.
T	T	On the flip, both coins turn up tails.

Obviously there are two possibilities of both coins showing like faces, namely, either two heads or two tails, and two possibilities showing unlike faces.

Now consider the analogous situation with two propositions, P and Q, any two propositions which can be judged true or false. The following table shows all possible combinations of "true" and "false" given these two propositions:

P	Q	
T	T	In this combination, both are true sentences.
T	F	In this combination, the first one is true and the second sentence is false.
F	T	In this combination, the first one is false and the second sentence is true.
F	F	In this combination, both the sentences are false.

We can use these possibilities for all combinations, whether using conjunctions or disjunctions, and later on, implications. This amounts, as it were, to a summary situation, which at a glance defines the decisions which we have made regarding the truth and falsity of conjunctions and disjunctions. Then, too, this will enable us to use these diagrams to extend to further understandings in regard to negations and implications.

CONJUNCTION

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

In order to understand the meaning or translation of these symbols into practical everyday language better, consider, for example, that:

P = The weather is sunny and Q = We will have a picnic.

Let us read the table for the conjunction:

P	Q	$P \wedge Q$	
T	T	T	The weather is sunny and we will have a picnic.
T	F	F	The weather is sunny and we will not have a picnic.
F	T	F	The weather is not sunny and we will have a picnic.
F	F	F	The weather is not sunny and we will not have a picnic.

Do the same for the disjunction.

4.7 Putting Ideas Together

Merely asking a sequence of questions does not necessarily bring into the child's focus the pattern of development. Rearranging, putting into a diagram or meaningful symbolic arrangements helps. Therefore, after the children do the disjunction truth table, make sure these are in a prominent place.

Now ask some "pop" questions which could have been asked before also but which usually, now that the total picture, in symbolic form, is visible, inspire some fast insightful answers.

1. If you have two propositions, how many possibilities of truth and falsity do you have?
2. If you have one proposition, how many possibilities?
3. When both propositions are true, what can you say about the conjunction? the disjunction?
4. When both propositions have opposite truth values, what can you say about the conjunction? the disjunction?
5. What are some similarities in regard to the truth value of a conjunction and disjunction when both propositions are true? when both propositions are false?
6. What happens to the conjunction when one statement is true and one is false?
7. What happens to the disjunction when one statement is true and one is false?
8. Can you generalize into one statement a brief summary of the truth values of a conjunction and disjunction?

There has been some controversy about the value of teaching truth tables in the elementary and secondary schools, but if the students learn these as a tool for further reasoning, there does not seem to be much of a problem. Symbols, diagrams, tables--all should be graphic techniques designed to simplify further learning or the recognition of patterns or structures. If their limitations are realized and if they are taught in conjunction with the regular materials, teachers would be aware that they are only tools.

If these objectives are ignored, the truth tables become merely another subject or topic in a contemporary orthodoxy in elementary mathematics and of doubtful value; but if these objectives are recognized and pursued consciously, then teaching about truth tables may contribute to children's ability to organize their materials, recognize patterns, and summarize various situations.

EXERCISE B

Make disjunctions and conjunctions out of each of the following.

1. The moon appears bright. The clouds are overcast.
2. A space traveler is an astronaut. The moon project is called Apollo.
3. Grass is green. Water is wet.
4. Judy is late for school. Her mother was not home.

Make conjunctions and disjunctions out of each of the following and determine whether each is true or false.

5. $5 \times 6 = 30$. $7 + 3 = 11$.
6. $3 \times (2 + 1) = (2 + 1) \times 3$. $19 - 18 = 19$.
7. 9 is a prime number. 13 is not divisible by three.
8. A square is a rectangle. Circles are round.

State whether each is a conjunction or a disjunction or neither. If possible, indicate the truth value of each.

9. If a number is even, then it has a factor of two.
10. Nine is between 8 and 10.
11. Six is greater than 5 or seven is less than 9.
12. Circles are a subset of the set of closed curves. Polygons are circles.
13. $\square = 10$ or $\square = 14$.
14. Beethoven was a musician or Alexander Hamilton was a flute player.
15. $7 + 1$ is an odd number or $3 + 2$ is less than $3 + 1$.
16. If Jones invented a dispenser for indispensable objects, then he is a genius.
17. Cats wear socks.
18. One is not the smallest prime number and two is the smallest prime number.
19. There are not many even prime numbers.
20. Triangles have fewer sides than quadrilaterals or pentagons have six sides.

5.1 The Negation "Not"

The negation of any statement, P, denoted by $\sim P$, represents the denial or opposite of P and will be written and read, "not P." That is, $\sim P$ denotes a statement which is true when P is false and false when P is true.

For example, the sentence "The 34th president of the U.S. was Dwight Eisenhower" may be symbolized:

D = The 34th president of the U.S. was Dwight Eisenhower.

The statement represented by D is true. Therefore, -D, or the sentence, "The 34th president of the U.S. was not Dwight Eisenhower," is false. Since D is true, -D, of necessity, is false.

Consider another example where B = The 15th president of the United States, James Buchanan, was a married man. Then -B = The 15th president of the United States, James Buchanan, was not a married man. Or -B could be translated, a bit more awkwardly, "It is false that the 15th president of the United States, James Buchanan, was a married man." Since B is false, -B must be true.

Consider a final example where we let S = There are 50 states in the United States. Then -S = There are not 50 states in the United States. Can both S and -S be true? It is impossible (and even strongly against our "common sense" notions) that a statement and its denial are both true. If S is true, then -S is false. If S is false, then -S must be true.

In truth table form, we have:

P	-P
T	F
F	T

It should be noted here, as it was when we were considering the connectives, <u>and</u>, <u>or</u>, that some words are used in ordinary language in slightly different context from what we do in formal logic or in mathematics at times. In everyday language, we commonly use <u>and</u> only when the two statements to be joined are relevant to each other in what might be called connected discourse. While advisable, it is not absolutely necessary. However, certain alternatives are available for those who stalk around using a "too disconnected a dialogue."

In logic, both formally and informally, the connectives, such as <u>and</u>, <u>or</u>, may be used with any two statements whatsoever, without regard to their possible relevance or bearing to one another as in the sentence, "Christmas is in December and 2 divided by 2 is one."

Most of the time we talk about related or connected ideas, but once again, it is imperative that we can distinguish between what is customary and what is absolutely essential. While it might not make much difference at this stage in the development of the child, it is imperative that teachers understand this distinction so that certain nonessentials are not over-emphasized or over-taught to the point of constituting a barrier to a later learning situation.

5.2 Contradiction

Intuitively, a contradiction means that something is not quite right, that there is an inherent lack of orderliness or "sensibleness" to a sequence of ideas or statements. Practically everyone when asked about this will agree when there is a lack of consistency in a sequence of statements. Therefore, it seems to be intuitive that in understanding what a contradiction is, more than one statement is necessary. It is somewhat like a relation; it demands at least two of something.

Suppose, for example, that Mary Jones is wearing a red coat. We make the true statement as follows:

(A) Mary Jones is wearing a red coat.

Now consider the following statements:

a) Mary's coat is brown.
b) Mary's coat is blue.
c) Mary's coat is green.
d) Mary's coat is not red.

Which one negates (A)?

The statement of $(A \wedge d)$ is a contradiction. That is, "Mary Jones is wearing a red coat and Mary Jones is not wearing a red coat," is a contradiction. The rest of the statements, (a), (b), (c), when combined with (A) are actually contraries, that is, they both cannot be true but both can be false. It is quite possible that both statements (A) and (a) could be false, but they both cannot be true. In the case of a contradiction, if one component is true, then necessarily, the other component must be false. Therefore, this requires the idea of negation, a denial of one of the statements.

When we speak of not as a connective, we have to be mindful of a slightly different situation from the other connectives such as and and or. The connective not is used with a single atomic sentence, that is, a simple statement.

Consider, for example, the sentence, "Dogs do not smoke pipes." This is not an atomic or simple statement even if it might resemble one because it contains only one simple sentence. It is the negation of "Dogs smoke pipes." Adding a connective not to an atomic sentence makes it a compound sentence in logic. The reason for this is implied, not expressed, in the sense that each negation carries with it some notion of an opposite. It is difficult to think of a sentence such as "Astronauts are not birds" without some idea of "Astronauts are birds."

This is a rather subtle point at this stage and it need not be a point of emphasis for the children in elementary mathematics. However, the teacher must be aware of this; and when she urges students to give opposites or denials, she should be careful to include the idea of "not" as carrying with it the implication of its affirmative.

5.3 Negation of conjunctions and disjunctions

We have agreed that:

A conjunction is true if and only if both sentences are true. In every other case, it is false.

A disjunction is true if at least one of the sentences is true. It is false if and only if both components are false.

Consider the following sentences:

1) A square is a rectangle and a triangle is a polygon.
2) A circle is a square or $2 + 2 = 4$.

Is the first sentence true? Obviously it is. Suppose that you want to negate it. What did you learn about a conjunction being false? (If one of the sentences is false, then the conjunction is false.) In order to deny the conjunction, do you think that we need to deny both of the sentences or just one? (Usually the students answer just one.) Which one? (I don't exactly know.) Therefore, it should be one or the other, shouldn't it? But which one?

Let us symbolize sentence (1):

S = A square is a rectangle.

T = A triangle is a polygon.

Symbolically, the sentence now reads: $S \wedge T$.

The negation of "$S \wedge T$" is symbolized: $-(S \wedge T)$. Putting the sentence back into English it reads awkwardly: "It is not true that a square is a rectangle and a triangle is a polygon." If the child were just given the sentence "$S \wedge T$" and told to negate it, perhaps it would be easier for him to realize that even though we do not know exactly what S means or T means, nevertheless we can still negate the conjunction.

What makes a conjunction false? If at least one of the components is false, the conjunction is false. You would not know which of the components, S or T, represents a false statement, so you can negate both of them and change the and to or, so that:

The negation of $S \wedge T$ or $-(S \wedge T)$ now becomes $-S \vee -T$.

Just introducing the sentence itself without the symbols has seemed artificial to most children and perhaps it is. We are trying to get a generalization that will include all negations of a conjunction but which seems to be superfluous in some individual cases. Many children will respond that the negation of "A square is a rectangle and a triangle is a polygon," is "A square is not a rectangle and a triangle is a polygon." When you examine this more closely, it does make sense; but negating the first component of a conjunction does not always negate the conjunction. (To see this, examine a conjunction with two false components.) Therefore, it would seem that the presentation of the negation of both conjunctions and disjunctions might better be done symbolically.

Consider the sentence:

3) Washington is the capital of the United States and Chicago is a western state.

This mixed-up geography can be negated without ever admitting lack of knowledge or determining whether the statement is true or false. Suppose that you did not know your geography and you considered sentence (3) to be true, but you want to negate it. So what do you do? You do the same as you did for the first example, namely, deny both of the sentences and change and to or. Its denial then becomes:

Washington is not the capital of the United States or Chicago is not a western state.

Here again, you might ask, "Why not just deny both components and keep the and, or just deny one of them and change the and to or?" Let us look at the sentences symbolically.

W = Washington is the capital of the United States, and

C = Chicago is a western state.

In symbolic form, the sentence is: $W \wedge C$. Its negation follows the same pattern as did the conjunction in sentence (1), namely, that $-(W \wedge C)$ is equivalent to $-W \vee -C$. To convince yourself that $-(W \wedge C)$ is equivalent to $-W \vee -C$, make a truth table for each statement letting W and C take on all possible truth values. You will see, if you do this, that $-(W \wedge C)$ and $-W \vee -C$ always have the same truth value for given values of W and C.

Let us now consider the negation of a disjunction like the one given in sentence (2): "A circle is a square or 2 + 2 = 4." Is the sentence true? Is a circle a square? Does 2 + 2 = 4? To deny it is to give it the opposite truth

value. The disjunction is true because $2 + 2 = 4$. In order to make it false, you must deny that the disjunction is true, and in order to make a disjunction false, both components must be false, so you write:

A circle is not a square and $2 + 2 \neq 4$.

The pattern for the negation of a disjunction is similar to that for the conjunction. In the latter case we negated both sentences and changed the <u>and</u> to <u>or</u>. For the disjunction we negate both sentences and change the <u>or</u> to <u>and</u>.

Consider another example:

4) Hockey is a winter sport or golf is a summer sport.

This is true and it so happens that both sentences are true. How can it be negated? Deny both sentences and change the <u>or</u> to <u>and</u>. The negation is:

Hockey is not a winter sport and golf is not a summer sport.

Now consider sentences (2) and (4) again, but written in symbolic form.

C = A circle is a square or	H = Hockey is a winter sport or
T = 2 + 2 = 4.	G = Golf is a summer sport.
$C \vee T$	$H \vee G$

Suppose you were just given the two disjunctions in symbolic form:

$C \vee T$ and $H \vee G$.

You cannot determine truth or falsity because you do not know what these symbols, C, T, H, and G, represent. But you can know that once you determine their meaning, you can fit your assertion or its denial into one of four possibilities.

What makes a disjunction false? Both components must be false in order to make the disjunction false so if you have a symbolic sentence such as $C \vee T$ or $H \vee G$, you will have to make both components false and change the <u>or</u> to <u>and</u>.

Therefore, the negation of $C \vee T$ becomes $-C \wedge -T$ and the negation of $H \vee G$, that is, $-(H \vee G)$ becomes $-H \wedge -G$. Actually you are negating the disjunctions and while you are not sure whether the original C, T, H, G, and their combining sentences, the disjunctions $C \vee T$ and $H \vee G$, are true or false, you are sure that you have changed their truth value. If you are skeptical, and you have a right to be, construct a truth table for $-(P \vee Q)$ and $-P \wedge -Q$, and you will see they always have the same truth value.

To determine the necessary flexibility, there should be a discussion with the children so that they really see that some of these equivalent expressions do mean the same things. The wave lengths of communication must be such that there is not only an impression that the expressions are the same but that the

reality is there; that both the children and teacher receive the same information or knowledge from a statement of equivalence. For example, the statement, "This problem is tough," should mean the same to both teacher and student, but in teen-age jargon, tough can sometimes mean "neat," "cool," "interesting," etc.

What is sometimes thought of as inconsistent thinking or "failure to understand" occurs because the wrong channels are turned on. Both parties are listening but to different statements. As the children get older, their vocabulary not only becomes more colorful, but the color is mixed; the hues become more subtle and what was "square" or "flakey" yesterday becomes something else today and tomorrow. It is necessary for all concerned to get the proper impression, whether it is "pop" communication which is similar to pop art in the sense that interpretation is a free-lance sort of thing or whether it is a prosaic, factual type of thing without any static.

Exercise C

Form the negation of each of the following and indicate the truth value of each original statement and its negation, whenever possible.

1. Mice are animals.
2. $2 + 3 = 5$.
3. $7 < 8$.
4. Bears are animals and June is a month of the year.
5. $x < 2$ and $4 + 8 = 7$.
6. Mice can read or dogs smoke cigars.
7. Milwaukee is a city and Christmas is in December.
8. $3 \div 2$ is less than 8 and 3 is the smallest odd prime.

Inferential Questioning Techniques Exemplified

9. $\frac{1}{6} + \frac{1}{7} > \frac{1}{8} + \frac{1}{9}$ and $x + 357 = 930$.

Despite the fact that fractions are involved in this example, a student who considered #5 carefully should be able to make some progress on #9. An inferential questioning sequence such as the following could be introduced here:

a) Is 1/6 greater than 1/8? If there is hesitation, use the flowchart idea--going backwards. In other words, the child has two choices. He knows or he does not know. If he knows, proceed; if he hesitates or doesn't know either by not saying a word or by making an error, ask the following:

i) Is 1 greater than 2?

ii) Is 1 greater than 1/2?

iii) Is 50¢ greater than 25¢?

iv) What is another name for 25¢?

v) What is another name for 50¢?

vi) Which would you rather have, a half-dollar or a quarter?

[Most children at the stage when fractions <u>and</u> money are both meaningful, will answer this correctly. At any rate, if you wish to reword this to the idea, "Which is worth more?" then you will be sure to get the correct answer. However, the odds of your getting a person in the middle elementary grades who does not like money are rather phenomenal and the odds increase with the age of the student.]

b) Is 1/7 greater than 1/9?

c) Is 1/6 greater than 1/8?

d) If you add 1/6 to 1/7, will it be greater than 1/8 + 1/9?

e) Is it necessary in this case to find a unique sum, that is, should you do the operation before you can answer the question of inequality?

10. Every triangle is a simple closed curve.

It is difficult to point out the idea of quantifiers, the universals, <u>all</u>, <u>no</u>, and the particular, <u>some</u>, but it is wise to ask some questions to determine what intuitive ideas the children have about these. From this knowledge, the teacher is better able to sharpen the precision or place emphasis on areas in which there has been a lack of experience or exposure. Questions such as those that follow could be used to investigate the children's conceptions of quantifiers.

a) Is every curve a closed curve?

b) Are all curves closed curves?

c) Are some curves closed curves?

d) Can you draw a picture of a curve?

e) Can you draw a picture of a closed curve?

f) Do all triangles have three sides?

g) Do some triangles have three sides?

h) Is it true that no triangles have four sides?

i) Are all simple closed curves triangles?

j) Are some simple closed curves triangles?

k) Are some simple closed curves circles?

l) Are some simple closed curves quadrilaterals?

11. N is greater than 7.

Teachers and students should be allowed to make equivalent expressions in these negations but there must be an assurance that there is equivalency. For example, to negate: "N is greater than 7," one can say, "N is not greater than 7." Equivalently to negate, one can also state: "N is less than or equal to 7." Frequently, children forget that if N is greater than 7, their assumption that "N is less than 7" as being the negation is only a kind of half-truth. There is another choice, "N could be equal to 7."

Some questions which might help to build this idea of a double-barreled negation are:

a) Is 8 greater than 7?

b) Is 5 greater than 7?

c) Is 7 greater than 7?

d) (Presuming the student answered "NO" to both questions (b) and (c)--) Does your "No" mean exactly the same thing in (b) and (c)?

e) How are they different? (As you know, if students do not give you the answer that builds on this sequence, ask some more detailed questions about the relation of other numbers and 7.)

f) Would you say that if a number is not greater than 7, that there are two choices?

g) What are they?

h) Can you indicate symbolically the negative of $n > 7$?

i) What would be the negative of $n < 7$?

j) What would be the negative of $n = 7$?

6.1 Universals and Particulars

A conversation does not get too far before someone uses such expressions as "some," "all," "every," "no," "none," "few," etc. These expressions put the statements into some kind of a quantitative category that tells us about everything in the category, a few things in the category, or nothing in the category.

"All children are noisy." "All teachers are mean." "Some boys are good at sports." "No one understands me." "Some teachers are not sympathetic." Statements similar to these five examples are legion in their availability. Actually few people mean what they say when they remark, "All children are noisy," because certain doctors will tell about youngsters brought to them because their parents felt that they were unnaturally quiet. So, we have a counterexample, and that is all we need to establish the falsity of a universal statement.

In other words, we have one statement or example that shows that the statement is not always true; it runs counter to what is supposed never to be false. If we were to assert that "All mathematics teachers are gorgeous," and really believed it, it would take only one person who was not a candidate for Miss America to illustrate the falsity of this statement. It should not be too difficult to find a counterexample!

Let us review briefly the two universal statements:

<u>Affirmative</u>: All A is B.

<u>Negative</u>: No A is B.

There are many examples of these in elementary mathematics such as the following:

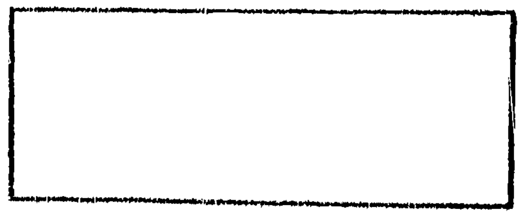
a) All squares are rectangles.
b) No odd number is an even number.
c) All even numbers are divisible by two.
d) No circles are polygons.

In the middle elementary as well as in the upper elementary and junior high grades, there are many references made to universal and particular statements. For example, if we read that "All prime numbers have only two distinct divisors," this means <u>all</u>. There is no exception. However, there is a condition--the numbers about which we are talking in this context are prime numbers. Often a universal statement using <u>all</u>, means all of a special or particular class. At times ambiguity is apt to exist, but try to avoid this by establishing a precision of language while admitting to the great need for flexibility. This amounts to flexible reasoning in a more precise way or precision in reasoning with a maximum of flexibility.

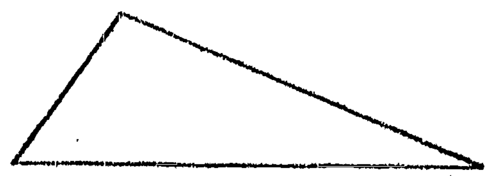
When we say that all squares are rectangles, we actually mean every last one of them. There is no exception to this. If you are able to find a square

that does not fit this pattern, you must re-analyze, get your glasses checked or start thinking about some new system of mathematics. The very structure of the geometric figure demands that this universal hold and that there is no exception.

It would not be correct, however, to reverse the statement and say, "All rectangles are squares." One need only to find a rectangle such as the one shown at the right. All sides of this figure are not congruent. Therefore, a counterexample is available, and we cannot accept the truth of this universal statement. The statement must be negated, but this is not done by excluding the entire group of rectangles from the picture. After all, only one rectangle was needed to show the universal statement false. Hence, intuition and formal logic agree: one counterexample is sufficient to show that a universal is not correct.



In mathematics and logic, some means at least one. Therefore, when the statement, "Some closed curves are polygons" is made, only one example need be found to show the truth of this. We are saying no more, no less than the fact that at least one closed curve that is a polygon can be found, and one is shown in the picture at the right.



Certainly there are more, but that is not the point at the moment. While discussing the meaning of the word some with children, it is essential that they recognize that sometimes the colloquial use and the mathematical use are not in absolute one-to-one correspondence. Some means different things to different people at different times. On occasion, some means three, few, many, five or six, or an indeterminate number, definitely not all. In mathematics, some simply means at least one. It means no more or less than that, but possibly it could mean all.

Take for example, the sentence: "Some teachers in this school are Democrats." Suppose, after investigation (nothing cloak-and-dagger, of course), you discover that all of the teachers in that school are Democrats. Your sentence does not become false according to the mathematical and logical interpretation. Colloquially, your confreres might not agree, but the mathematical meaning allows for expansion--that is, if we begin with too little information, we can use some and will not have to change the sentence if it just happens to be all.

This does not mean that a student has to become confused about the double meaning of this word <u>some</u>. As he matures, he begins to learn that the context of a word requires some adjustment in interpretation. In the same sense, a lawyer uses words in a legal fashion; nevertheless, most of them are able to carry on casual conversations with their non-lawyer neighbors. The difficulty is to keep the neighborliness out of the language when used in a legal or mathematical sense.

7.1 The Implication "If . . ., Then . . ."

Teachers are prone to use such statements as "If you behave, then you will have an extra reading period," or "If it does not rain tomorrow, then we will have our ball game." Children seem to get the message but when it comes down to a careful analysis, things are not quite so simple.

Suppose someone says to you, "If I earn $20, then I will take you to the circus." You become quite happy; and if you want to go to the circus, you begin to think this is a <u>sure thing</u>, despite the "if" component which places the condition of earning $20 on being taken to the circus. Your friend has not really promised you that he will take you to the circus. There is a condition, a requirement here, namely, that something else must happen--$20 must be earned.

Before going into a careful analysis of the <u>if . . . then</u> statements or implications, as they are called, it is well to face reality and indicate that the definition and interpretation of implication are among the most controversial topics in formal and symbolic logic. In order not to get embroiled in the argument, we shall take a rather intuitive, "everyday point of view" outlook that should be helpful in teaching.

There are several ways of looking at conditional sentences (implications), and while one wants to sharpen his precision, nevertheless there must always be an awareness of the difference between concise logical implications and colloquial conditions. Utter frustration could betake anyone who unrealistically presumes that all persons use the "<u>if. . . then . . .</u>" statements the way a logician or mathematician interprets them.

7.2 Some Basic Notions

"<u>If . . . then . . .</u>" is one of the logical connectives. Thus a proposition containing "<u>if . . . then . . .</u>" is a compound proposition and connects two statements. In contrast to the relatively simple connectives, <u>and</u>, <u>or</u>, we must be

sure to note that there are two connectives involved in an implication. Occasionally in everyday usage, the then will not be stated formally, but it is implicit. For example, the sentence, "If Mary leaves school early, then she will miss play practice" is an implication. The sentence could also be written, "If Mary leaves school early, she will miss play practice," omitting the "then."

As in the case of the other connectives, we can write an implication using symbols. If M = Mary leaves school early and P = She will miss play practice, then "$M \longrightarrow P$" (to be read "M implies P" or "If M, then P.") is the symbolic representation of the sentence "If Mary leaves school early, then she will miss play practice."

Consider the sentence, "If the team wins, then it will play in the championship game." Let T = The team wins and P = It will play in the championship game. Then "$T \longrightarrow P$" is the symbolic form of the sentence. The sentence following the "if" is called the antecedent or hypothesis; that following "then" is the consequent or conclusion.

The two propositions connected by "If . . . then . . ." do not need to have any causal relation or dependence on each other. In ordinary language, and in mathematical and logical language, for that matter, "If . . . then . . ." (and also and, or) commonly join two propositions that are relevant to each other. However, what is usually so and what is necessarily so are not always synonymous. Nonsensically, we could write, "If Christmas is in June, then Rip Van Winkle will buy a new car." Then we could proceed to analyze it bit by bit but not much would be accomplished except a little fun.

7.3 Different Kinds of Implications

An implication (hypothetical statement or conditional—whichever name you wish) does not say that the antecedent is true but only that if its antecedent is true, then its consequent is true also. It does not state that its consequent is true but only that its consequent is true if its antecedent is true. There is a relation of implication between the if-clause and the then-clause.

When disjunctions were studied, it was learned that there is more than one meaning of "or." So it is with an implication—it also has more than one meaning. Perhaps the best way to examine the meanings is to list some examples which illustrate different kinds of implication, or a different sense of "if . . . then . . ."

1) If all three-sided polygons are triangles and figure 1 is a three-sided polygon, then figure 1 is a triangle.
2) If seven has only two distinct divisors, then seven is a prime number.
3) If litmus paper is placed in acid, then it will turn red.
4) If Michigan loses the football game, then he will lose his bet.

Only a casual examination of the four examples indicates the different uses. In (1) the consequent follows LOGICALLY from the antecedent. The statement forms a complete LOGICAL ARGUMENT whereby the word "then" has the force of a "therefore."

In (2) the consequent follows by the very definition of the word "prime," so we call this conditional DEFINITIONAL, that is, the consequent follows as an explication of the nature of the antecedent. The consequent of (3) does not follow by definition or logically from its antecedent; the connection or relation must be discovered empirically and this conditional is CAUSAL, that is, the antecedent expresses the cause and the consequent expresses the effect.

In (4) the consequent represents a state of affairs, as it were, which the person intends to bring about (happily or not) if the antecedent becomes a reality. This is called a DECISIONAL conditional. While the four examples might seem to be completely different, they are all IMPLICATIONS and as such there must be a common, pervading idea in each of them.

7.4 Truth Values of Implications

It was necessary to make a decision about the truth and falsity of disjunctions after we examined carefully the possible meanings. So also must we look at the possibilities with an implication. Recall that when we have two propositions, we have four possible combinations of truth values, such as:

P	Q
T	T
T	F
F	T
F	F

Each component sentence has a truth value and the composition, in this case, the implication, has a truth value also. An important question is: What

circumstances establish the falsehood of a given implication? Each of the four examples examined previously would agree in this also.

Let us use the example: If the litmus paper is placed in acid, then it will turn red. Symbolize the statement by letting P = The litmus paper is placed in acid and Q = It will turn red. Let us compare the possible choices in the original examination of the possibilities of two propositions with some English translations.

P	Q	
T	T	1. Litmus paper is placed in acid. Litmus paper turns red.
T	F	2. Litmus paper is placed in acid. Litmus paper does not turn red.
F	T	3. Litmus paper is not placed in acid. Litmus paper does turn red.
F	F	4. Litmus paper is not placed in acid. Litmus paper does not turn red.

Which of the conditionals is true; which false? (1) presents no difficulty. We can agree that (1) as a conditional is true. In (2) the paper is placed in acid but it does not turn red. Our circumstances are not "right," as the antecedent is true, the consequent is false. We say then that the conditional or implication is false.

Note that the original implication by itself makes no strong statement about litmus paper in acid or turning red. It only states that if the litmus paper is placed in acid, then it will turn red. It makes no claim about what would occur if it were not placed in acid. Therefore, we can agree that the implication s (3) and (4) are true. It could just be that if the litmus paper were not placed in acid, it might turn red and it might not turn red.

It seems futile to try to justify the truth values assigned to (3) and (4). Students might prefer other choices of truth values, but the logical system remains consistent with recognized standard inference forms and is in "tune" with other logical ideas. Therefore, regardless of whether the conditional states a logical connection between its antecedent and consequent, a definitional connection, a causal or decisional connection, if the antecedent is true and the consequent is false, then the implication itself is false.

The following truth table defines the truth value of the implication for all values of the components in a compact form:

P	Q	P ⟶ Q
T	T	T
T	F	F
F	T	T
F	F	T

One other situation is possible regarding the "connection" between the antecedent and consequent, namely, the case when there is no real connection. An example is the sentence: "If tea grows in Thailand, then there is four feet of snow in Detroit." When this occurs, it is called material implication. Although in elementary mathematics it seldom occurs that we are concerned with material implication, it is necessary for teachers to realize that such things do exist (for better or for worse). It is another case of a distinction between what is usually so and what is absolutely true. In other words, most common sense people show some connection or relationship between their antecedents and consequents in implications, but it is not absolutely necessary.

8.1 Concluding Remarks

In an attempt to develop some of the ideas of logical thinking, we have tried to illustrate the necessity of strengthening intuition while at the same time realizing its limitation. There is no dichotomy between logical thinking or reasoning and the "hunch" or intuitive approach. Each has its place, complementing the other in the development of the child's mathematical literacy.

Of major importance in discussing the logical and intuitive aspects of thinking with children is the ability of the teacher to guide the child and to lead him by judicious questioning to think through the reasoning process. Furthermore, both components in this process, the teacher and the student, must not only pay attention to vocabulary but must also realize that words take on different meanings when used colloquially and technically.

Relating much of the development of reasoning ability to the ordinary experiences of the child and guiding him to see their parallel in many applications of mathematics is an important job for the elementary teacher. Hopefully she will, by sharpening her own thinking abilities, help the student to attain his own reasoning potential more efficiently.